CHAPTER 9

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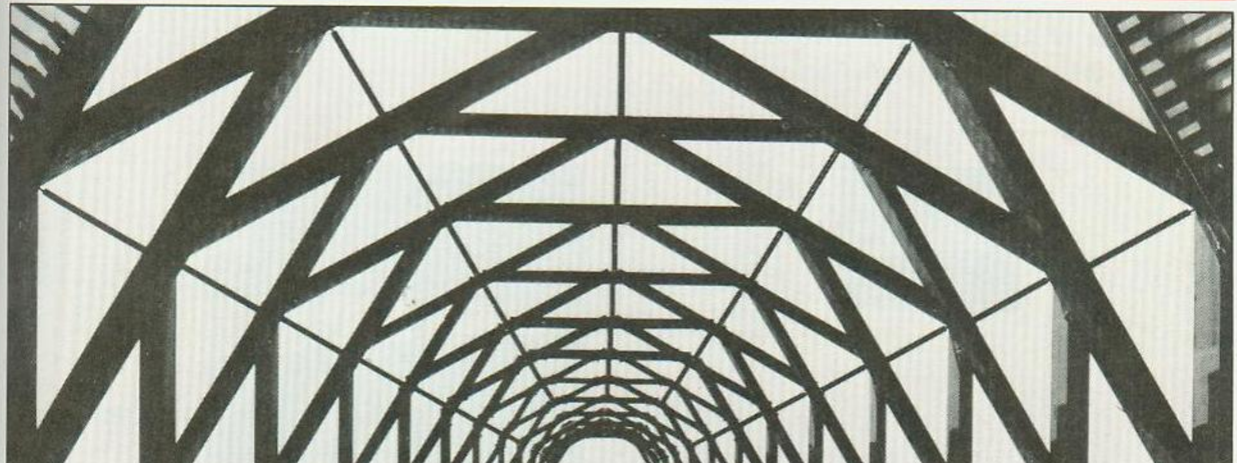
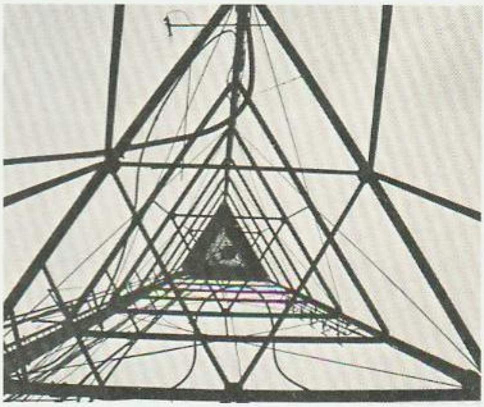
Similarity



9–1 Proportions

In this chapter you will study polygons that have the same shape—but not necessarily the same size. There are many triangles in this photograph that have the same shape.



The idea of same shape is involved in an enlargement or a reduction. Consider the two photographs below.

Both of these photographs are of the same subject, but one is larger than the other. Both pictures have the "same shape." If we compare the ratio of width to length of each picture, we see that the ratios are equal.

$$\frac{4\text{ cm}}{5\text{ cm}} = \frac{8\text{ cm}}{10\text{ cm}}$$

This equation is called a *proportion* because it is made up of two equal ratios, $\frac{4}{5}$ and $\frac{8}{10}$.

Definition 9-1

A **proportion** is an equality between two ratios. The ratios $\frac{a}{b}$ and $\frac{c}{d}$ are proportional if $\frac{a}{b} = \frac{c}{d}$. $b \neq 0$, $d \neq 0$.

(It is important to remember that a denominator cannot be equal to zero.)

In this lesson we shall review some of the algebraic properties of proportion. The proofs of these theorems will be omitted, but each will be preceded by a numerical example.

Example 1 In a proportion cross products are equal.

$$\frac{4}{8} \overset{\searrow}{\underset{\nearrow}{=}} \frac{2}{4} \quad \text{or} \quad 4 \times 4 = 2 \times 8$$

Theorem 9–1 If $\frac{a}{b} = \frac{c}{d}$, then $a \times d = b \times c$.

Example 2 In a proportion, 1 can be added to both sides.

If $\frac{9}{3} = \frac{12}{4}$, then $\frac{9}{3} + \frac{3}{3} = \frac{12}{4} + \frac{4}{4}$ or $\frac{9+3}{3} = \frac{12+4}{4}$.

Theorem 9–2 If $\frac{a}{b} = \frac{c}{d}$, then $\frac{a+b}{b} = \frac{c+d}{d}$.

Example 3 In a proportion, 1 can be subtracted from both sides.

If $\frac{9}{3} = \frac{12}{4}$, then $\frac{9}{3} - \frac{3}{3} = \frac{12}{4} - \frac{4}{4}$ or $\frac{9-3}{3} = \frac{12-4}{4}$.

Theorem 9–3 If $\frac{a}{b} = \frac{c}{d}$, then $\frac{a-b}{b} = \frac{c-d}{d}$.

Example 4 If $\frac{9}{3} = \frac{12}{4}$, then $\frac{9}{12} = \frac{3}{4}$.

Theorem 9–4 If $\frac{a}{b} = \frac{c}{d}$, then $\frac{a}{c} = \frac{b}{d}$.

Example 5 If $9 \times 4 = 3 \times 12$, then $\frac{9}{3} = \frac{12}{4}$.

Theorem 9–5 If $a \times d = b \times c$, then $\frac{a}{b} = \frac{c}{d}$.

EXERCISES

A.

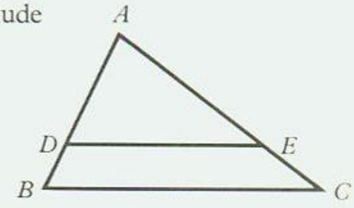
1. Given that $\frac{4}{5} = \frac{12}{15}$, which theorem is used to conclude each of the following:

   a. $\frac{4}{12} = \frac{5}{15}$ **b.** $\frac{9}{5} = \frac{27}{15}$ **c.** $\frac{-1}{5} = \frac{-3}{15}$.

2. Suppose $\frac{AD}{DB} = \frac{AE}{EC}$. Which theorem is used to conclude each of the following:

   a. $\frac{AD - DB}{DB} = \frac{AE - EC}{EC}$ **b.** $\frac{AD}{AE} = \frac{DB}{EC}$

   c. $\frac{AD + DB}{DB} = \frac{AE + EC}{EC}$.



3. Use Theorem 9-5 and the following products to form proportions.

   a. $3 \times 4 = 2 \times 6$ **b.** $\sqrt{2} \times \sqrt{3} = 1 \times \sqrt{6}$

   c. $2 \cdot MN = 3 \cdot XY$ **d.** $AB \cdot CD = EF \cdot GH$

4. Use Theorem 9-1 to solve the following proportions for x.

   a. $\frac{4}{5} = \frac{x}{25}$ **b.** $\frac{6}{9} = \frac{18}{x}$ **c.** $\frac{x - 4}{x} = \frac{3}{4}$

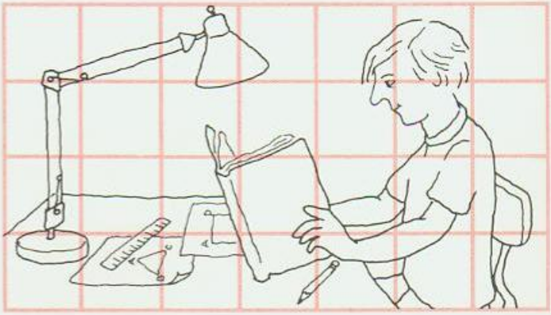
B.

5. State the contrapositive of Theorem 9-1. Use this contrapositive to show that the following statements are not proportions.

   a. $\frac{12}{13} = \frac{13}{14}$ **b.** $\frac{23}{33} = \frac{31}{41}$ **c.** $\frac{\sqrt{2}}{\sqrt{3}} = \frac{\sqrt{2} + 1}{\sqrt{3} + 1}$

Activity

A 1-cm square grid has been superimposed on this cartoon. Draw an enlargement of this cartoon by using a 2-cm square grid.



6. The measures of two complementary angles are in a ratio of $\frac{2}{3}$. Find the measures of these angles.

7. If 2 calculators cost $28, how much should 5 calculators cost?

8. Two numbers are in a ratio of $2:3$. What is the ratio of their squares?

9. The measures of two supplementary angles are in a ratio of $\frac{3}{5}$. Find the measures of the angles.

10. A 56 cm segment is divided into a ratio of 3 to 5. Find the length of the two segments.

11. A 5×7 photograph is enlarged by a factor of $\frac{5}{2}$. What size is the enlargement?

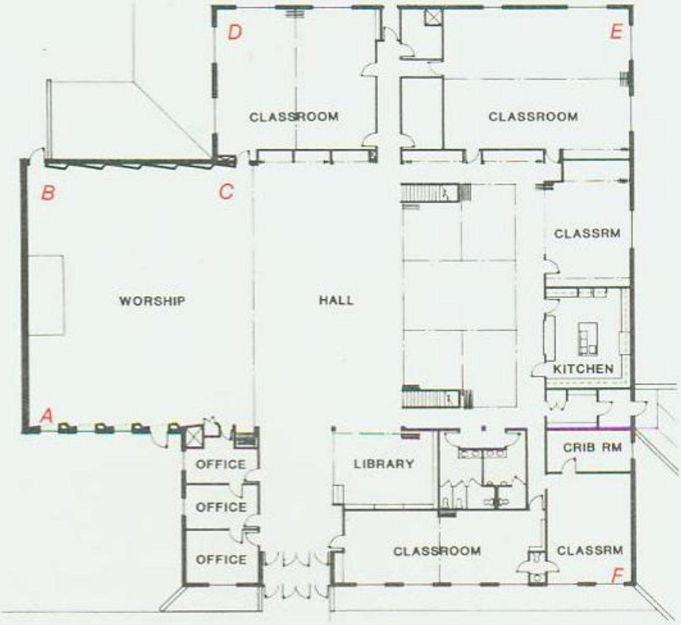
12. The area of two triangles are in a ratio of 4 to 9. The smaller triangle has an area of 50 cm^2. Find the area of the larger triangle.

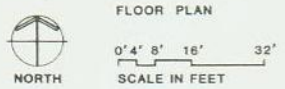
C.

13. Solve the proportion for x. $\frac{-3}{x} = \frac{x-8}{5}$

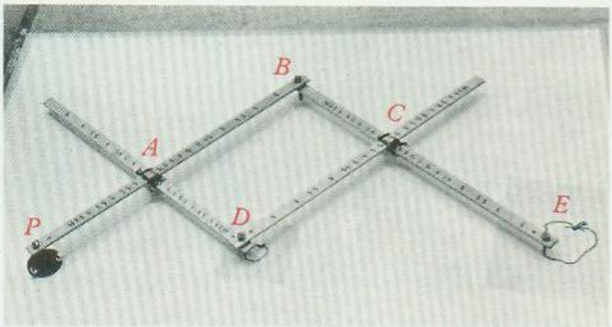
PROBLEM SOLVING

1. Use a ruler and the scale indicated below to find the actual distance across this church both east-west and north-south.
2. Write down the proportions used in answering Problem 1.
3. If the walls from A, north around the church to F, are poured concrete walls that cost $20/foot, estimate the total cost of pouring these walls.





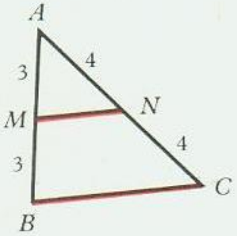
9–2 The Side-Splitting Theorem



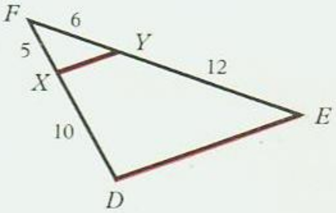
A mechanical linkage called a pantograph can be used by draftsmen and others to enlarge or reduce drawings, as shown here. The instrument lies flat and is fastened down at a pivot point P. As the stylus at point D traces a figure, the pencil at point E draws an enlarged figure. The pantograph is based upon the theorem in this lesson.

In the triangles below a segment has been drawn parallel to one side of the triangle.

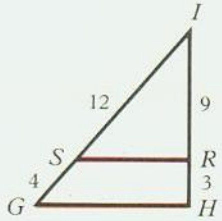
$\overline{MN} \parallel \overline{BC}$



$\overline{XY} \parallel \overline{DE}$



$\overline{SR} \parallel \overline{GH}$



Observe that

$$\frac{AM}{MB} = 1, \frac{AN}{NC} = 1, \qquad \frac{FX}{XD} = \frac{1}{2}, \frac{FY}{YE} = \frac{1}{2}, \qquad \frac{IS}{SG} = \frac{3}{1}, \frac{IR}{RH} = \frac{3}{1},$$

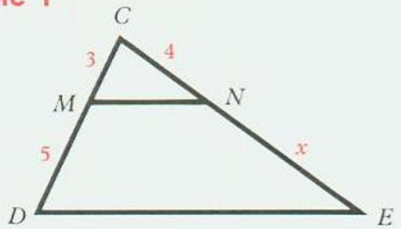
or or or

$$\frac{AM}{MB} = \frac{AN}{NC}. \qquad \frac{FX}{XD} = \frac{FY}{YE}. \qquad \frac{IS}{SG} = \frac{IR}{RH}.$$

These observations are summarized by this theorem.

Theorem 9–6 **Side-Splitting Theorem.** If a line is parallel to one side of a triangle and intersects the other two sides, then it divides the two sides proportionally.

Example 1



In this figure $\overline{MN} \parallel \overline{DE}$.
Find NE.

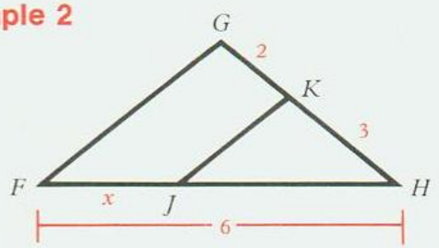
By Theorem 9–6, $\dfrac{CM}{MD} = \dfrac{CN}{NE}$,

hence $\dfrac{3}{5} = \dfrac{4}{x}$,

$$3x = 20,$$

or $x = \dfrac{20}{3}$.

Example 2



Find FJ in the figure shown if $\overline{GF} \parallel \overline{KJ}$.

By Theorem 9–6, $\dfrac{GK}{KH} = \dfrac{FJ}{JH}$,

$$\frac{2}{3} = \frac{x}{6 - x},$$

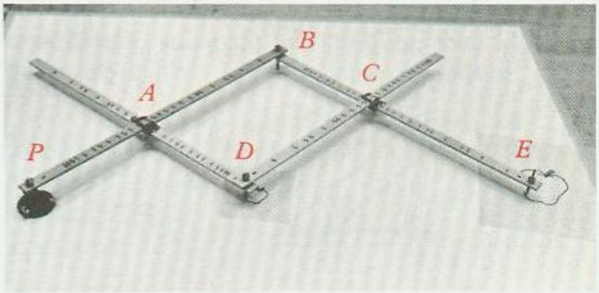
$$3x = 12 - 2x,$$

$$5x = 12,$$

or $x = \dfrac{12}{5}$.

APPLICATION

1. A pantograph is constructed so that $AB = CD$, $AD = BC$, and P, D, and E are collinear. Therefore by Theorem 8–4 we can conclude that $ABCD$ is a parallelogram.
2. Since $ABCD$ is a parallelogram, $\overline{AD} \parallel \overline{BE}$ in $\triangle PBE$. Therefore, by Theorem 9–6, we can conclude that the ratio $\dfrac{PD}{DE}$ is always constant and equal to the ratio $\dfrac{PA}{AB}$.



This fact helps explain why the pantograph works.

The converse of Theorem 9–6 is also true and is stated here.

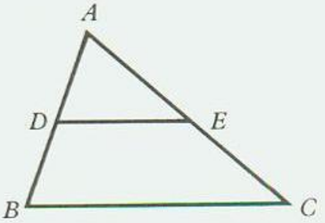
Theorem 9–7 If a line intersects two sides of a triangle and divides them proportionally, then the line is parallel to the third side.

EXERCISES

A.

1. Assume $\overline{DE} \parallel \overline{BC}$. Indicate whether the following are true or false.

   a. $\frac{AD}{DB} = \frac{AE}{EC}$
   b. $\frac{AB}{AD} = \frac{AC}{AE}$
   c. $\frac{AB}{DB} = \frac{AC}{EC}$
   d. $\frac{AD}{AE} = \frac{EC}{BD}$
   e. $\frac{AD}{AE} = \frac{AB}{AC}$
   f. $\frac{AD}{AB - AD} = \frac{AE}{AC - EC}$



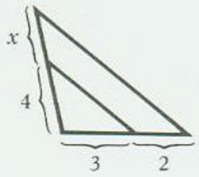
(Exs. 1, 2)

2. Indicate whether the following are true or false.

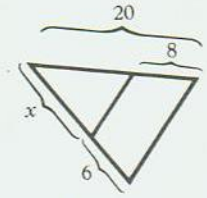
   a. If $\frac{DB}{AD} = \frac{EC}{AE}$, then $\overline{DE} \parallel \overline{BC}$.
   b. If $\frac{DB}{AE} = \frac{EC}{AD}$, then $\overline{DE} \parallel \overline{BC}$.
   c. If $\frac{AB}{AD} = \frac{AC}{AE}$, then $\overline{DE} \parallel \overline{BC}$.
   d. If $\frac{AD}{AB} = \frac{EC}{AC}$, then $\overline{DE} \parallel \overline{BC}$.
   e. If $\frac{DB}{AB - DB} = \frac{EC}{AC - EC}$, then $\overline{DE} \parallel \overline{BC}$.
   f. If $\frac{AD + DB}{AD} = \frac{AE + EC}{AE}$, then $\overline{DE} \parallel \overline{BC}$.

In exercises 3–8, find the value of x. Assume the lines that appear parallel are parallel.

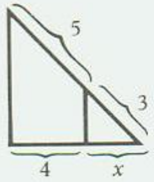
3.



4.



5.



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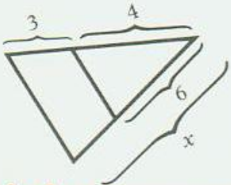

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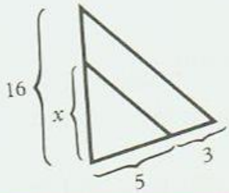
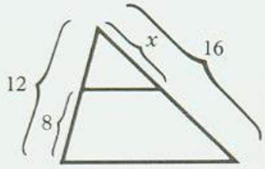


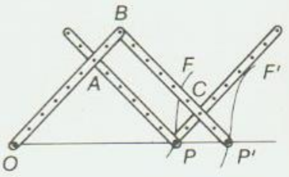
8.



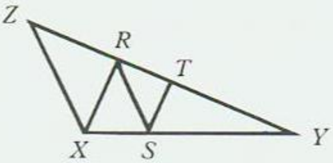
Activity

For this activity you will need strips of cardboard or wood about 30 cm long and some fasteners.

1. Make a pantograph by fastening strips together so that $AB = PC$ and $AP = BC$.
2. Use it to draw an enlargement of some figure.

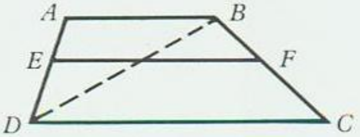


B.



9. Given: $\overline{RS} \parallel \overline{ZX}$, $\overline{RX} \parallel \overline{TS}$

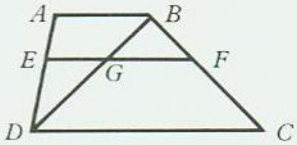
Prove: $\frac{TY}{RT} = \frac{RY}{RZ}$



10. Given: $ABCD$ is a trapezoid.
$\overline{EF} \parallel \overline{AB}$, $\overline{EF} \parallel \overline{DC}$

Prove: $\frac{AE}{ED} = \frac{BF}{FC}$

(*Hint:* Draw $\overline{BD}$. Consider $\triangle ABD$ and $\triangle BCD$.)



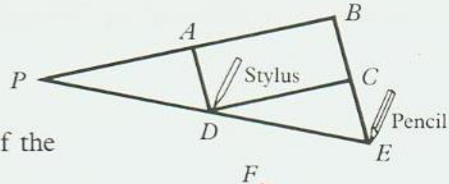
(Exs. 11, 12)

11. Given: $ABCD$ is a trapezoid.
$\overline{EF} \parallel \overline{AB}$, $\overline{EF} \parallel \overline{DC}$,
$\frac{AE}{ED} = \frac{1}{4}$, $BC = 30$

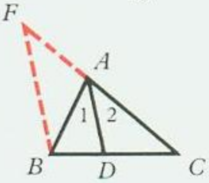
Find: BF and FC

12. Given: $ABCD$ is a trapezoid.
$\overline{EF} \parallel \overline{AB}$, $\overline{EF} \parallel \overline{DC}$,
$\frac{BG}{GD} = \frac{3}{4}$, $AD = 8$, $BC = 12$

Find: AE, ED, BF, FC



13. In this pantograph, $PA = 8$ cm and $AB = 24$ cm. If the stylus at D traces a segment of 14 cm, what is the length of the segment drawn by the pencil at E?



(Exs. 14, 15)

14. Given: $\triangle ABC$, $m\angle 1 = m\angle 2$

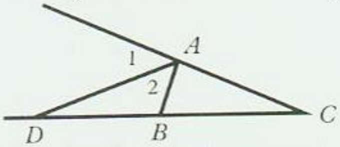
Prove: $\frac{BD}{DC} = \frac{AB}{AC}$

(*Hint:* Construct $\overline{FB} \parallel \overline{AD}$.)

15. Given: $\overline{AD}$ bisects $\angle BAC$, $AB = 10$,
$AC = 15$, $BC = 18$

Find: BD and DC

(*Hint:* See exercise 14.)



16. Given: $\triangle ABC$, $m\angle 1 = m\angle 2$

Prove: $\frac{BD}{DC} = \frac{AB}{AC}$

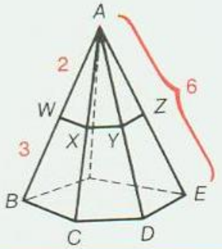
(*Hint:* Use the Side-Splitting Theorem and an auxiliary line.)

PROBLEM SOLVING

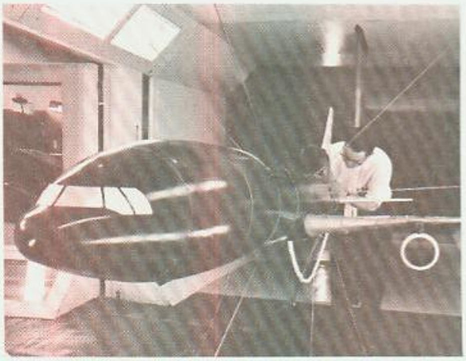
In this pentagon-based pyramid:

$\overline{WX} \parallel \overline{BC}$, $\overline{XY} \parallel \overline{CD}$, $\overline{YZ} \parallel \overline{DE}$, and $AW = 2$, $WB = 3$, $AE = 6$

Find AZ and ZE.



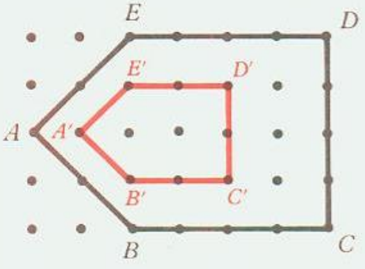
9–3 Similar Polygons



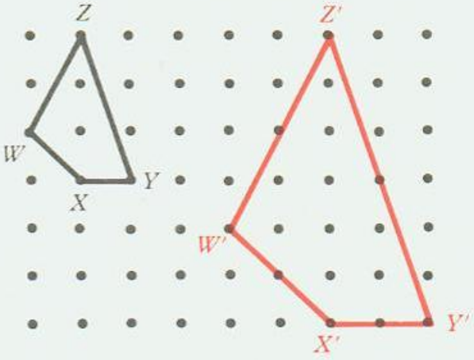
Industrial designers often build models of projects that will eventually be constructed full size. The model of the plane has the same shape as the actual aircraft.

In this lesson we shall focus our attention on polygons, and we shall describe what is meant by *identical in shape,* or *similar*. Consider these figures.

ABCDE is similar to *A′B′C′D′E′*.



WXYZ is similar to *W′X′Y′Z′*.



$\angle A \cong \angle A'$, $\angle B \cong \angle B'$, $\angle C \cong \angle C'$
$\angle D \cong \angle D'$, $\angle E \cong \angle E'$

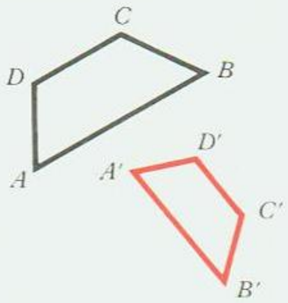
$$\frac{AB}{A'B'} = \frac{BC}{B'C'} = \frac{CD}{C'D'} = \frac{DE}{D'E'} = \frac{AE}{A'E'} = 2$$

$\angle W \cong \angle W'$, $\angle X \cong \angle X'$,
$\angle Y \cong \angle Y'$, $\angle Z \cong \angle Z'$

$$\frac{WX}{W'X'} = \frac{XY}{X'Y'} = \frac{YZ}{Y'Z'} = \frac{WZ}{W'Z'} = \frac{1}{2}$$

Notice that all corresponding angles are congruent and that the ratios of corresponding sides are equal.

The symbol "∼" means "is similar to."
$ABCD \sim A'B'C'D'$ means that $ABCD$ is similar to $A'B'C'D'$.

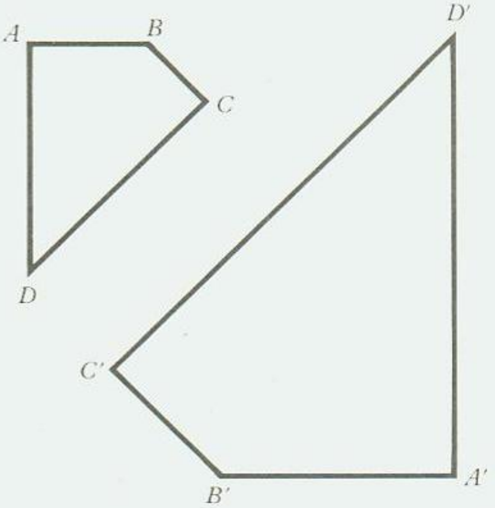


Definition 9–2

Two polygons are **similar** if there is a correspondence between vertices such that corresponding angles are congruent and corresponding sides are proportional.

Example 1 If we are given that $ABCD \sim A'B'C'D'$, then we can conclude:

1. $\angle A \cong \angle A'$, $\angle B \cong \angle B'$, $\angle C \cong \angle C'$, $\angle D \cong \angle D'$.
2. $\dfrac{AB}{A'B'} = \dfrac{BC}{B'C'} = \dfrac{CD}{C'D'} = \dfrac{AD}{A'D'}$.



Example 2 If we are given that

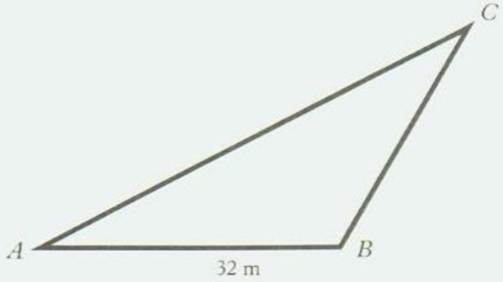
1. $\angle A \cong \angle A'$, $\angle B \cong \angle B'$, $\angle C \cong \angle C'$, $\angle D \cong \angle D'$ and
2. $\dfrac{AB}{A'B'} = \dfrac{BC}{B'C'} = \dfrac{CD}{C'D'} = \dfrac{AD}{A'D'}$,

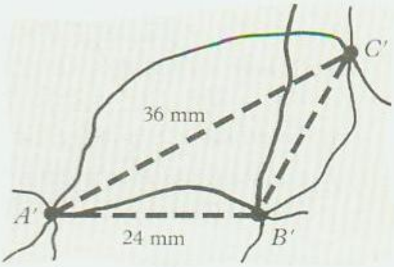
then we can conclude that $ABCD \sim A'B'C'D'$.

APPLICATION

If you have a scaled map, then a figure on the map is similar to the figure it represents. Given a scaled map of locations A, B, and C, then $\triangle ABC \sim \triangle A'B'C'$.

If $A'C' = 36$ mm, $A'B' = 24$ mm, and $AB = 32$ m, find AC.





Solution: $\dfrac{AB}{A'B'} = \dfrac{AC}{A'C'}$ or $\dfrac{32\text{ m}}{24\text{ mm}} = \dfrac{AC}{36\text{ mm}}$.

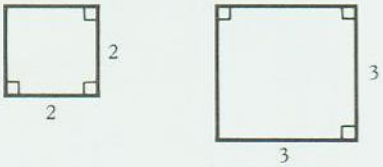
Therefore $AC = \dfrac{32\text{ m} \times 36\text{ mm}}{24\text{ mm}} = 48$ m.

EXERCISES

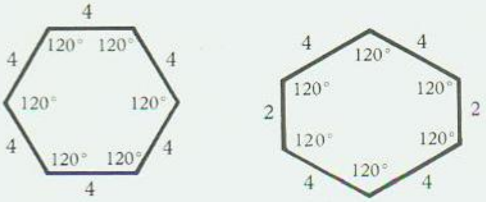
A.

For exercises 1-4, decide whether or not the given pair of polygons are similar.

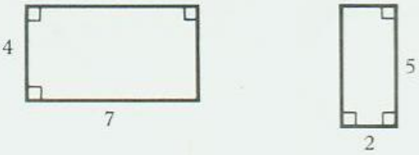
1.



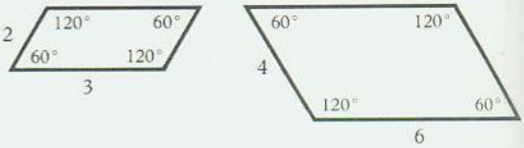
2.



3.



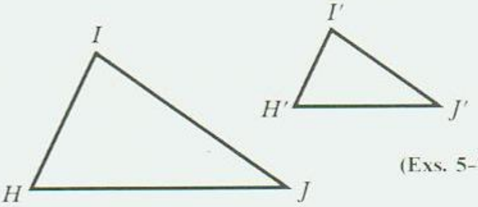
4.



5. Suppose $\triangle HIJ \sim \triangle H'I'J'$. Name three pairs of corresponding angles that are congruent.

6. If $\triangle HIJ \sim \triangle H'I'J'$, complete this proportion in two different ways.

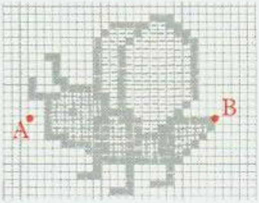
$$\frac{HI}{H'I'} = \underline{\ ?\ }$$



(Exs. 5-7)

7. Suppose $\triangle HIJ \sim \triangle H'I'J'$, and that $H'I' = 4$, $I'J' = 6$, $H'J' = 7$, and $HJ = 12$. Find lengths HI and IJ.

8. Using the scale indicated on this needlepoint pattern, find the length AB in the enlarged finished product.



Scale: 5 mm = 2 cm

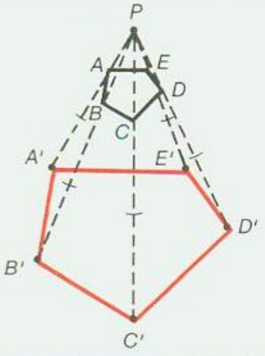
Activity

Here is a method of constructing a polygon similar to *ABCDE*.

a. Select a point *P* and draw $\overrightarrow{PA}$, $\overrightarrow{PB}$, $\overrightarrow{PC}$, $\overrightarrow{PD}$, and $\overrightarrow{PE}$.

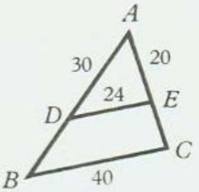
b. Construct points A', B', C', D', E' as shown so that $AA' = 2PA$, $BB' = 2PB$, $CC' = 2PC$, $DD' = 2PD$, and $EE' = 2PE$.

1. Trace *ABCDE* and try the construction stated above.
2. Locate point *P* in a different position and repeat the construction.



B.

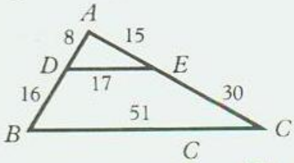
In this figure $\overline{DE} \parallel \overline{BC}$.



(Exs. 9, 10)

9. If $\dfrac{AD}{AB} = \dfrac{AE}{AC} = \dfrac{DE}{BC}$, prove that $\triangle ADE \sim \triangle ABC$.

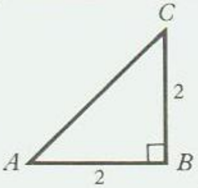
10. If the length of segments are as shown, then find DB and EC. (*Hint:* Use exercise 9.)



(Ex. 11)

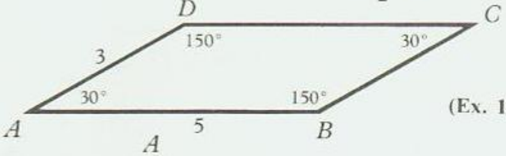
11. If $\overline{DE} \parallel \overline{BC}$, and the lengths of segments are as shown, prove that $\triangle ADE \sim \triangle ABC$.

12. The lengths of the sides of a pentagon are 6, 8, 9, 12, and 15. If a similar pentagon has a longest side of 4, find the lengths of the remaining sides.



(Ex. 13)

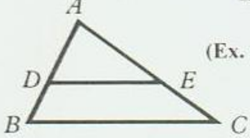
13. Construct a right triangle similar to $\triangle ABC$ with a leg of length $\sqrt{2}$.



(Ex. 14)

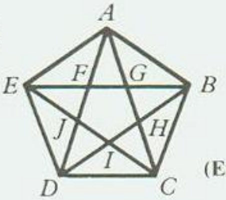
14. Construct a quadrilateral similar to $ABCD$, whose shortest side has length 9.

C.



(Ex. 15)

15. Suppose $\triangle ADE \sim \triangle ABC$. Prove that $\dfrac{DB}{AD} = \dfrac{EC}{AE}$. (This is the conclusion of the Side-Splitting Theorem.)

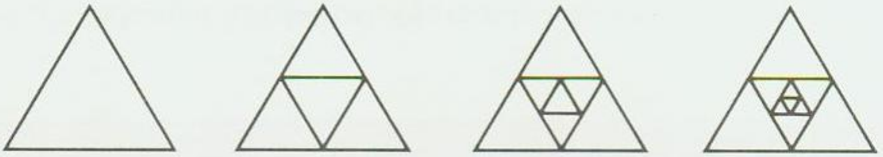


(Ex. 16)

16. Given that $ABCDE$ is a regular pentagon, and that $\dfrac{BD}{DC} = \dfrac{BC}{DI} = \dfrac{DC}{CI}$, state several pairs of similar triangles. Justify your choices.

PROBLEM SOLVING

Use the problem solving method "Make a Table" (page 167) to solve this problem.



no subdivisions — 1 triangle

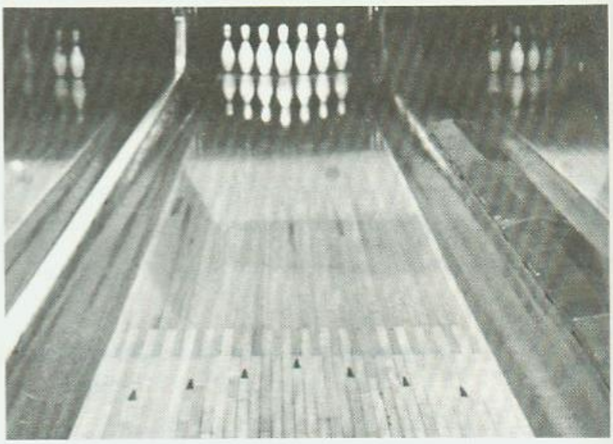
one subdivision — 5 triangles

two subdivisions — 9 triangles

three subdivisions — ? triangles

If the equilateral triangle were subdivided ten times, how many triangles would result?

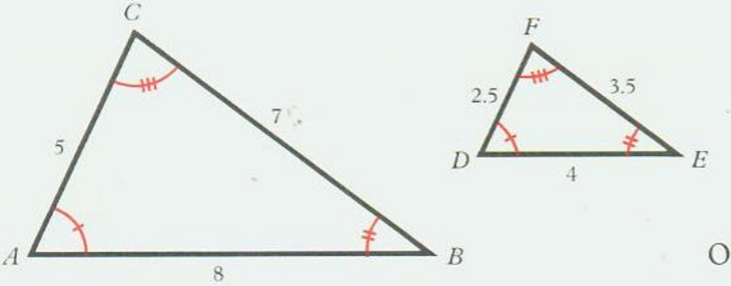
9–4 The AAA Similarity Postulate



In the sport of bowling, the bowler uses the sight marks to aim the ball. Suppose a bowler aims for the second mark and misses it by 2 centimeters. By how much will the ball miss the pin?

This question can be answered by applying the theorem studied in this lesson.

In $\triangle ABC$ and $\triangle DEF$, $\angle A \cong \angle D$, $\angle B \cong \angle E$, and $\angle C \cong \angle F$.



Observe that $\dfrac{AB}{DE} = \dfrac{BC}{EF} = \dfrac{CA}{FD}$.

It appears that whenever all three angles of one triangle are congruent to all three angles of another triangle, then the ratios of corresponding sides are also equal. We accept this as a postulate.

AAA Similarity Postulate

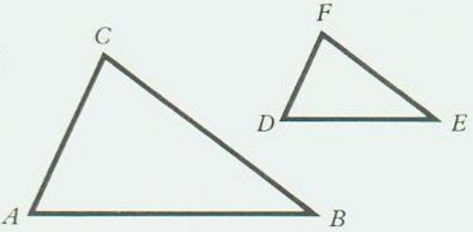
If three angles of one triangle are congruent to three angles of another triangle, then the triangles are similar.

The next theorem describes a simple method for proving two triangles similar.

Theorem 9–8 **AA Similarity Theorem.** If two angles of one triangle are congruent to two angles of another triangle, then the triangles are similar.

PROOF

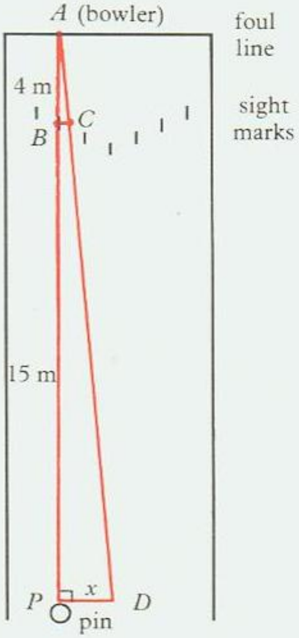
Given: $\triangle ABC$ and $\triangle DEF$ with $\angle A \cong \angle D$, $\angle B \cong \angle E$
Prove: $\triangle ABC$ is similar to $\triangle DEF$.



Statements	Reasons
1. $\angle A \cong \angle D$	1. Given
2. $\angle B \cong \angle E$	2. Given
3. $\angle C \cong \angle F$	3. Why ?
4. $\triangle ABC \sim \triangle DEF$	4. AAA Similarity

APPLICATION

When the bowler misses the mark by 2 cm, the ball misses the pin by how much? Consider triangles $\triangle ABC$ and $\triangle APD$. These triangles are constructed to be right triangles, and they have a common angle A. Therefore by Theorem 9-8, we conclude that $\triangle ABC \sim \triangle APD$.



$$\frac{AB}{AP} = \frac{BC}{PD} \quad \text{or} \quad \frac{4\text{ m}}{19\text{ m}} = \frac{2\text{ cm}}{x\text{ cm}}$$

$$\text{or} \quad x = \frac{38}{4}\text{ cm} = 9\frac{1}{2}\text{ cm}$$

Applying the AA Similarity Theorem to the case of right triangles gives this theorem.

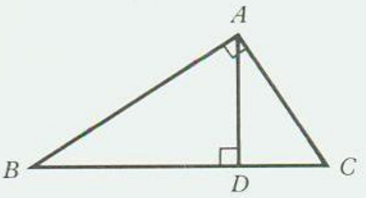
Theorem 9–9 Two right triangles are similar if an acute angle of one triangle is congruent to an acute angle of the other triangle.

EXERCISES

A.

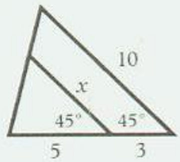
For exercises 1-4 $\triangle ABC$ and $\triangle XYZ$ are given. Complete the statement $\triangle ABC \sim$ _?_.

1. $m\angle A = 17$, $m\angle C = 49$, $m\angle X = 17$, $m\angle Z = 49$
2. $m\angle A = 23$, $m\angle B = 111$, $m\angle Y = 23$, $m\angle Z = 111$
3. $m\angle B = 68$, $m\angle C = 21$, $m\angle X = 21$, $m\angle Y = 91$
4. $m\angle C = 119$, $m\angle A = 24$, $m\angle X = 24$, $m\angle Y = 37$
5. Explain why $\triangle ABC \sim \triangle DBA$.

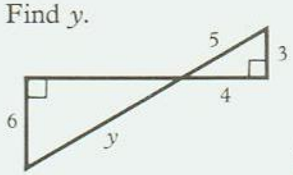


(Ex. 5)

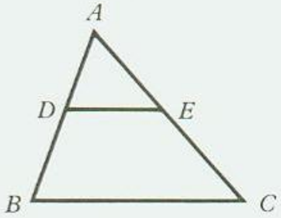
6. Find x.



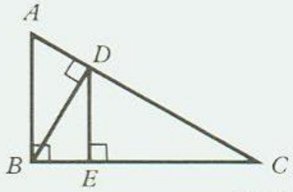
7. Find y.



8. If a 6-foot man has a shadow of 9 feet, how long a shadow would a 20-foot pole have?
9. If $\overline{DE} \parallel \overline{BC}$, $AD = 3$, $AB = 8$, and $BC = 9$, what is the length of $\overline{DE}$?



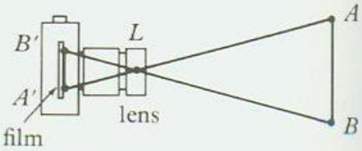
10. Are all isosceles right triangles similar? Why?
11. Are all right triangles similar? Why?
12. If two isosceles triangles have their vertex angles congruent, are they similar? Why?
13. Are all 30°-60°-90° triangles similar? Why?
14. List as many similar triangles as you can in the figure shown.



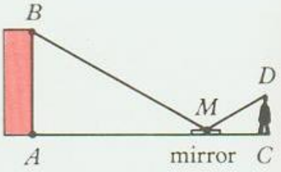
(Ex. 14)

B.

15. When a photograph is taken, the image formed on the film is similar to the object being photographed. Similar triangles help to explain this. If $\overline{AB}$ and $\overline{A'B'}$ are parallel, prove that $\triangle LAB$ and $\triangle LA'B'$ are similar.



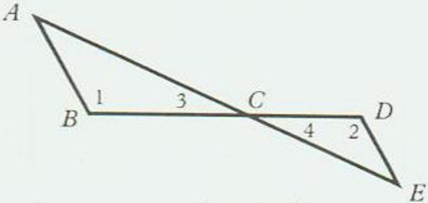
16. One method of finding the height of an object is to place a mirror on the ground and then position yourself so that the top of the object can be seen in the mirror. How high is a tower if a 150 cm tall person observes the top of the tower when the mirror is 120 m from the tower and the person is 6 m from the mirror?



$\angle AMB \cong \angle CMD$

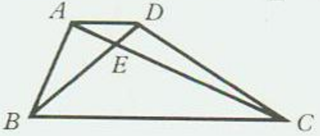
17. **Given:** $m\angle 1 = m\angle 2$
Prove: $\triangle ABC \sim \triangle EDC$

18. **Given:** $\overline{AB} \parallel \overline{DE}$
Prove: a. $\triangle ABC \sim \triangle EDC$
b. $\frac{AC}{CE} = \frac{AB}{DE}$



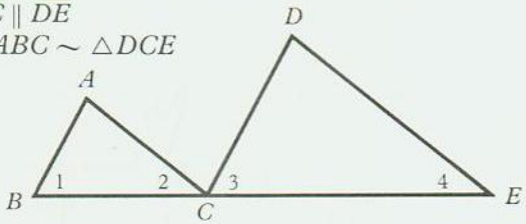
(Exs. 17, 18)

19. **Given:** $ABCD$ is a trapezoid
Prove: $\triangle AED \sim \triangle CEB$



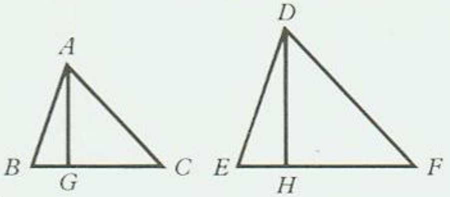
(Ex. 19)

20. **Given:** $\overline{AB} \parallel \overline{DC}$
$\overline{AC} \parallel \overline{DE}$
Prove: $\triangle ABC \sim \triangle DCE$



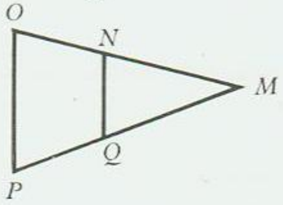
(Ex. 20)

21. **Given:** $\triangle ABC \sim \triangle DEF$
$\overline{AG}$ and $\overline{DH}$ are altitudes.
Prove: $\frac{AB}{DE} = \frac{AG}{DH}$



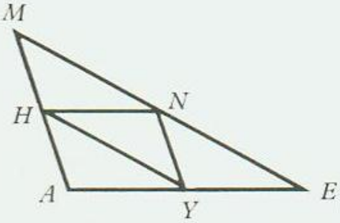
(Ex. 21)

22. **Given:** $\overline{NQ} \parallel \overline{OP}$
Prove: a. $\triangle MNQ \sim \triangle MOP$
b. $\frac{MN}{MO} = \frac{NQ}{OP}$



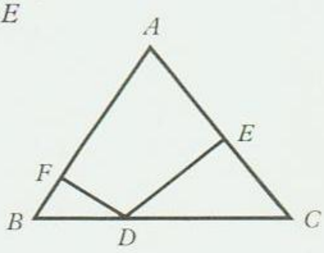
(Ex. 22)

23. **Given:** H, N, and Y are midpoints of $\overline{MA}$, $\overline{ME}$, and $\overline{AE}$.
Prove: $\triangle HNY \sim \triangle EAM$

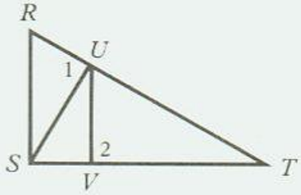


(Ex. 23)

24. **Given:** $\angle B \cong \angle C$, $\overline{DF} \perp \overline{AB}$, $\overline{DE} \perp \overline{AC}$
Prove: a. $\triangle BDF \sim \triangle CDE$
b. $\frac{FD}{ED} = \frac{BD}{CD}$

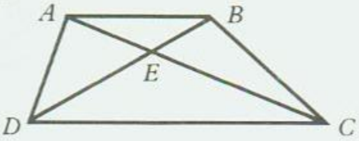


25. **Given:** $\angle RST$, $\angle 1$, and $\angle 2$ are right angles.
Prove: **a.** $\triangle RSU \sim \triangle RTS$
b. $\triangle UVT \sim \triangle RUS$



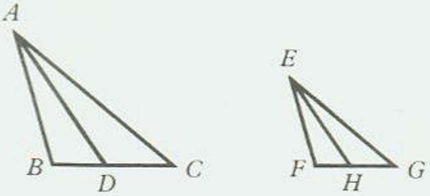
C.

26. **Given:** $ABCD$ is a trapezoid.
Prove: $AE \cdot DE = BE \cdot CE$

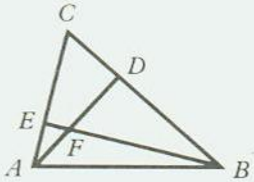


27. **Given:** $\triangle ABC \sim \triangle EFG$
$\overline{AD}$ bisects $\angle A$.
$\overline{EH}$ bisects $\angle E$.

Prove: $\frac{AB}{EF} = \frac{AD}{EH}$



28. **Given:** $\overline{AD} \perp \overline{BC}$, $\overline{BE} \perp \overline{AC}$
Prove: **a.** $\frac{AD}{BE} = \frac{AC}{BC}$
b. $AD \cdot BC = AC \cdot BE$



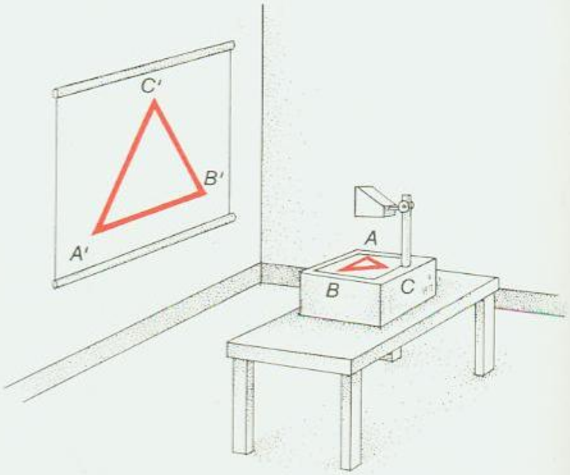
Activity

Set an overhead projector 10 feet from a screen and perpendicular to it. Place a cutout triangle ABC on the projector. Call the triangle on the screen $A'B'C'$.

1. Measure $\angle A$ and its image $\angle A'$. How do they compare?
2. Measure lengths AB and $A'B'$, AC and $A'C'$, BC and $B'C'$.

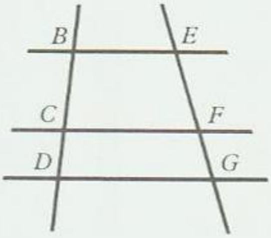
   How do the ratios $\frac{AB}{A'B'}$, $\frac{AC}{A'C'}$, and $\frac{BC}{B'C'}$ compare?

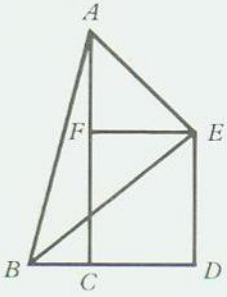
What do you conclude about $\triangle ABC$ and $\triangle A'B'C'$?



29. Given: $\overline{BE} \parallel \overline{CF} \parallel \overline{DG}$

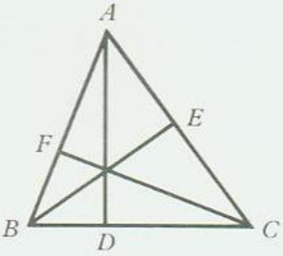
Prove: $\frac{BC}{CD} = \frac{EF}{FG}$





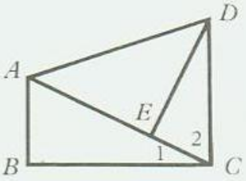
30. Given: $\overline{AC} \perp \overline{FE}$, $\overline{AC} \perp \overline{BD}$,
$\overline{DE} \perp \overline{BD}$, $\overline{AE} \perp \overline{BE}$

Prove: $\triangle AFE \sim \triangle BDE$



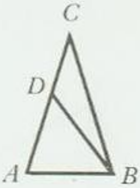
31. Given: $\overline{AD} \perp \overline{BC}$, $\overline{FC} \perp \overline{AB}$
$\overline{BE} \perp \overline{AC}$

Prove: $\frac{AF}{BF} \cdot \frac{BD}{CD} \cdot \frac{CE}{AE} = 1$



32. Given: $\overline{DE} \perp \overline{AC}$
$\overline{AB} \perp \overline{BC}$
$m\angle 1 + m\angle 2 = 90$

Prove: $BC \cdot CE = ED \cdot AB$



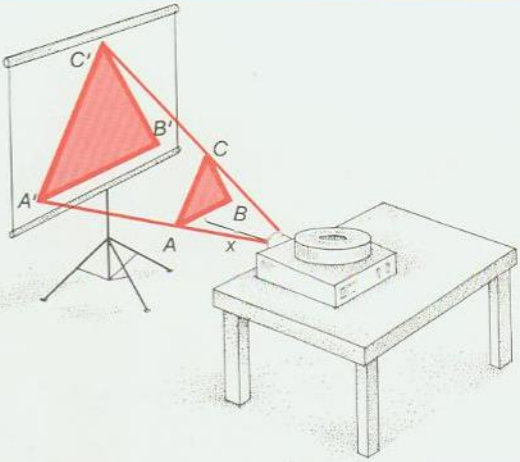
33. Given: $\overline{CA} \cong \overline{CB}$
$\overline{BA} \cong \overline{BD}$

Prove: $(AB)^2 = AC \cdot AD$

PROBLEM SOLVING

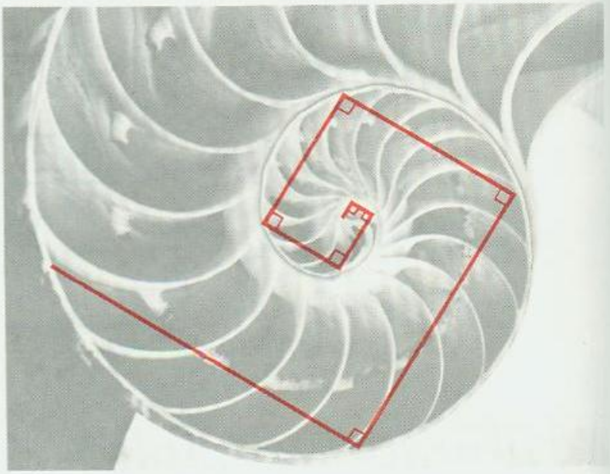
Suppose a slide projector and screen are set up as shown with the screen 20 feet from the projector. Assume $\triangle ABC$ is similar to $\triangle A'B'C'$.

1. If the triangular cutout $\triangle ABC$ is placed x feet in front of the projector, calculate the length $A'B'$ in terms of the length AB and the distance x.
2. If x is halved, what happens to $A'B'$?



9–5 Right Triangles and Similar Triangles

An interesting example of right triangle similarity in nature is the shell of the chambered nautilus. The photograph shows a shell that has been cut in half to reveal its spiral construction. This spiral can be approximated by a sequence of segments at right angles to one another as shown here. This spiral is related to the theorem of this lesson.



We begin with an example and a definition.
8 is the geometric mean between 4 and 16, since

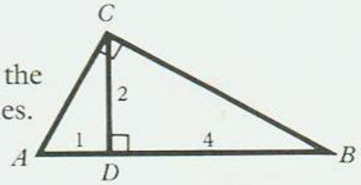
$$\frac{4}{8} = \frac{8}{16}.$$

Definition 9-3

A number x is a **geometric mean** between two numbers a and b if

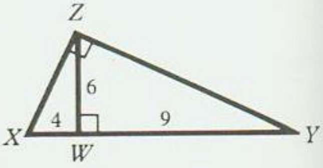
$$\frac{a}{x} = \frac{x}{b},\ x \neq 0,\ b \neq 0.$$

The idea of geometric mean is used in the next theorem. Consider these right triangles.



Observe that

$$\frac{AD}{CD} = \frac{CD}{DB}.$$



Observe that

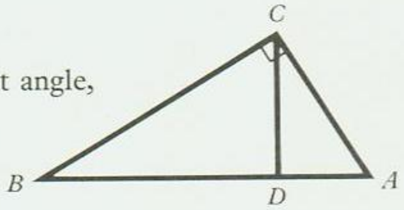
$$\frac{XW}{WZ} = \frac{WZ}{WY}.$$

Theorem 9–10 In a right triangle, the length of the altitude to the hypotenuse is the geometric mean between the lengths of the two segments of the hypotenuse.

PROOF

Given: $\triangle ABC$ with $\angle C$ a right angle, $\overline{CD}$ an altitude

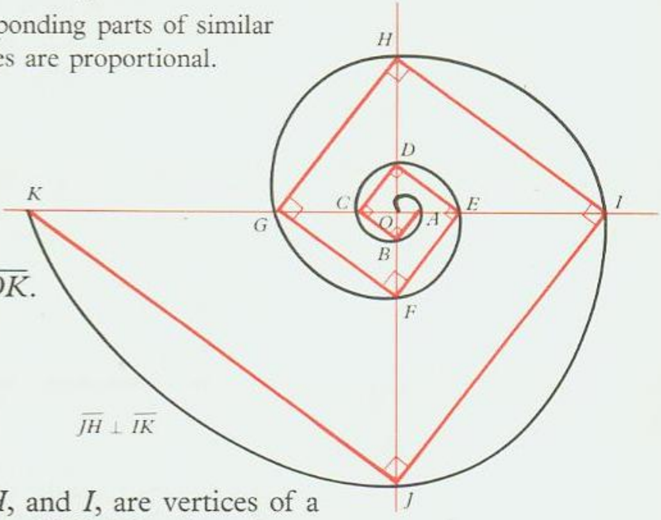
Prove: $\frac{AD}{DC} = \frac{DC}{DB}$



Statements	Reasons
1. $\angle ADC$ is a right angle.	1. $\overline{CD}$ is an altitude.
2. $\angle BDC$ is a right angle.	2. Why ?
3. $\angle C$ is a right angle.	3. Given
4. $\angle BCD$ is complementary to $\angle ACD$.	4. Why ?
5. $\angle CAD$ is complementary to $\angle ACD$.	5. Why ?
6. $\angle BCD \cong \angle CAD$.	6. Why ?
7. $\triangle ADC \sim \triangle CDB$.	7. Two right triangles are similar if an acute angle of one is congruent to an acute angle of the other.
8. $\frac{AD}{DC} = \frac{DC}{DB}$	8. Corresponding parts of similar triangles are proportional.

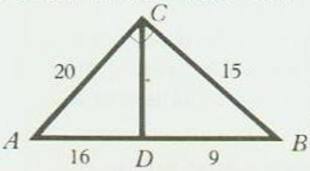
APPLICATION

The shell of the chambered nautilus is based on a geometric mean. Consider the sequence of radii
$\overline{OA}, \overline{OB}, \overline{OC}, \overline{OD}, \overline{OE}, \overline{OF}, \overline{OG}, \overline{OH}, \overline{OI}, \overline{OJ}, \overline{OK}$.
The length of each of these segments is a geometric mean between the lengths of the preceding segment and the succeeding segment.



Each three successive points, for example, G, H, and I, are vertices of a right triangle. Furthermore, $\overline{OH}$ is an altitude of $\triangle GHI$. Therefore, by Theorem 9–10, OH is the geometric mean between OG and OI.

This triangle illustrates Theorem 9–11. The proof of Theorem 9–11 is left as an exercise.



Observe that $\frac{AD}{AC} = \frac{AC}{AB}$ and $\frac{DB}{BC} = \frac{BC}{AB}$.

Theorem 9–11 Given a right triangle and the altitude to the hypotenuse, each leg is the geometric mean between the length of the hypotenuse and the length of the segment of the hypotenuse adjacent to the leg.

EXERCISES

A.

What is the geometric mean between the numbers in exercises 1–4?

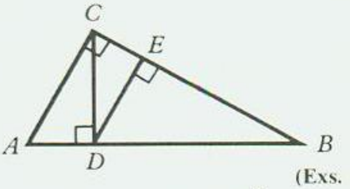
1. 4 and 9 **2.** 9 and 16 **3.** 4 and 5 **4.** $\sqrt{3}$ and $\sqrt{5}$

5. The length of $\overline{CD}$ is the geometric mean between the lengths of which two segments?

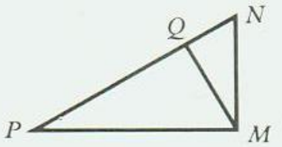
6. The length of $\overline{DE}$ is the geometric mean between the lengths of which two segments?

7. The length of $\overline{AC}$ is the geometric mean between the lengths of which two segments? (See Theorem 9-11.)

8. The length of $\overline{BC}$ is the geometric mean between the lengths of which two segments?



(Exs. 5-8)



(Exs. 9-16)

For exercises 9–16, use this figure in which $\angle PMN$ is a right angle and $\overline{MQ} \perp \overline{PN}$.

9. $PQ = 9$, $QN = 4$. Find MQ.

10. $QN = 3$, $MQ = 9$. Find PQ.

11. $PM = 12$, $PQ = 9$. Find PN.

12. $MN = 8$, $QN = 6$. Find PN.

13. $PN = 75$, $PQ = 72$. Find MN.

14. $MQ = 4$, $PN = 10$. Find QN.

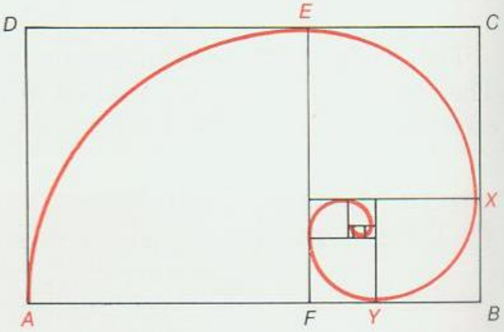
15. $PN = 13$, $PM = 12$. Find MQ.

16. $PM = 16$, $MN = 12$. Find PQ.

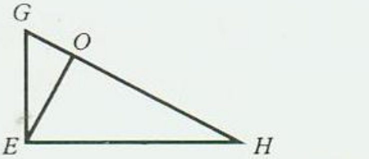
Activity

A spiral similar to the one found in the chambered nautilus shell is based on the golden rectangle. It can be constructed with compass and straightedge.

1. Construct a golden rectangle *ABCD* as shown in Geometry in Our World, page 301.
2. Subdivide the rectangle *BCEF* into a square and a rectangle, and continue subdividing the newly obtained rectangle into a square and a rectangle.
3. Construct in each square a circular arc as shown. The center of each circular arc is a vertex of the square.



B.

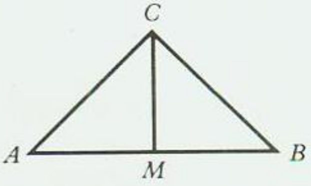


(Exs. 17-19)

17. Suppose $m\angle HEG = 90$ and $\overline{EO} \perp \overline{HG}$. If $HO = 6$ and $EG = 4$, find OG.

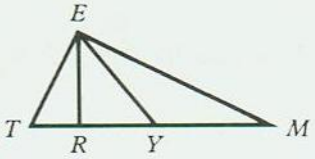
18. Suppose $m\angle HEG = 90$, $\overline{EO} \perp \overline{HG}$, $EO = 8$ and $\dfrac{HO}{OG} = \dfrac{2}{1}$. Find HO.

19. Suppose $m\angle HEG = 90$, $\overline{EO} \perp \overline{HG}$, $HO = 10$ and $OG = 8$. Find $HE \cdot EG$.

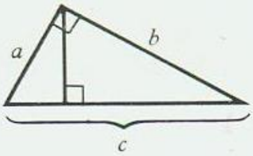


20. **Given:** $m\angle ACB = 90$, $\overline{CM} \perp \overline{AB}$, $AB \cdot CM = (AC)^2$
Prove: $AC = BC$

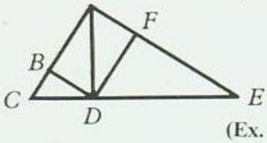
C.



21. **Given:** $m\angle TEM = 90$, $\overline{RE} \perp \overline{TM}$,
$TY = YM$, $ER = 4\frac{4}{5}$, $TE = 6$
Find: RY



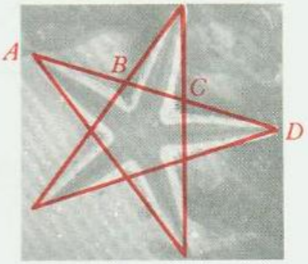
22. Use Theorem 9–11 and the figure to show that $a^2 + b^2 = c^2$ (Pythagorean Theorem).



(Ex. 23)

23. Suppose $AFDB$ is a rectangle and $\overline{AD} \perp \overline{CE}$. Show that the area of $AFDB = \sqrt{BC \cdot BA \cdot AF \cdot FE}$.

24. Prove Theorem 9–11.



$AB = CD$

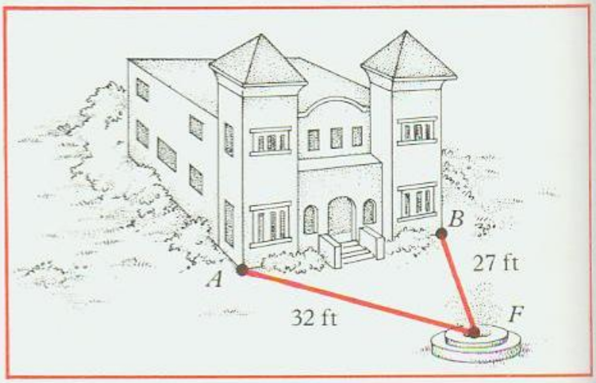
25. The geometric mean can be found in starfish in several ways. For example, in the five-pointed star shown here, AB is the geometric mean between BC and AC. Accept this fact and use it to show that AC is also the geometric mean between AB and AD. (*Hint*: Use Theorem 9–2.)

PROBLEM SOLVING

Show that the spiral constructed in the last activity is a spiral of the chambered nautilus.

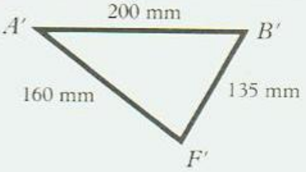
That is, show that *EX* is the geometric mean between *AE* and *XY*. (*Hint:* Use the definition of golden rectangle found in the Architecture Geometry in Our World, page 300, together with the fact that *ABCD* and *BCEF* are both golden rectangles.)

9–6 The SSS and SAS Similarity Theorems



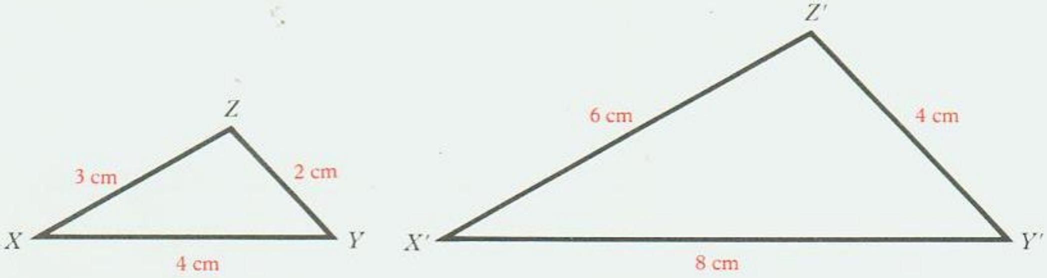
A water fountain is to be installed 32 feet from one corner of a building and 27 feet from another corner of the building. The building is 40 feet wide.

A set of working drawings for this project gives a scale of 5 mm to a foot. After the corners A' and B' of this building are located on the drawing, the point F' is drawn 160 mm from A' (5×32) and 135 mm from B' (5×27). Is $\triangle ABF$ similar to $\triangle A'B'F'$?



In the next example $\triangle XYZ$ and $\triangle X'Y'Z'$ are drawn so that

$$\frac{XY}{X'Y'} = \frac{YZ}{Y'Z'} = \frac{XZ}{X'Z'}.$$



When the sides of the triangles are drawn proportionally then, $m\angle X = m\angle X' = 30$, $m\angle Y = m\angle Y' = 46$, and $m\angle Z = m\angle Z' = 104$.

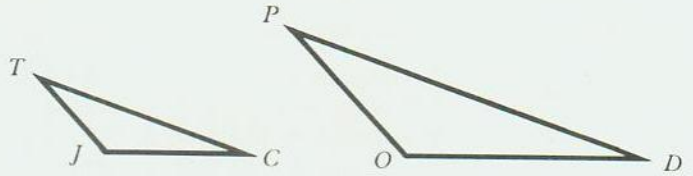
These examples suggest the theorem called the SSS Similarity Theorem.

Theorem 9–12 **SSS Similarity Theorem.** If three sides of one triangle are proportional to the three sides of another triangle, then the triangles are similar.

Theorem 9-12 says

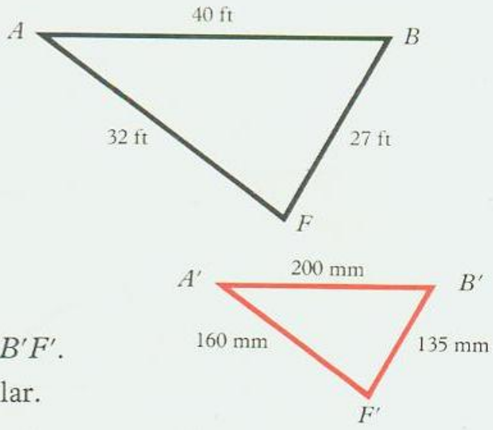
if $\frac{TJ}{PO} = \frac{JC}{OD} = \frac{TC}{PD}$,

then $\triangle TJC \sim \triangle POD$.



APPLICATION

In the example at the beginning of the lesson there is a triangle with sides of length 27 feet, 32 feet, and 40 feet, and a scale drawing with sides of length 135 mm, 160 mm, and 200 mm. Are these triangles similar?

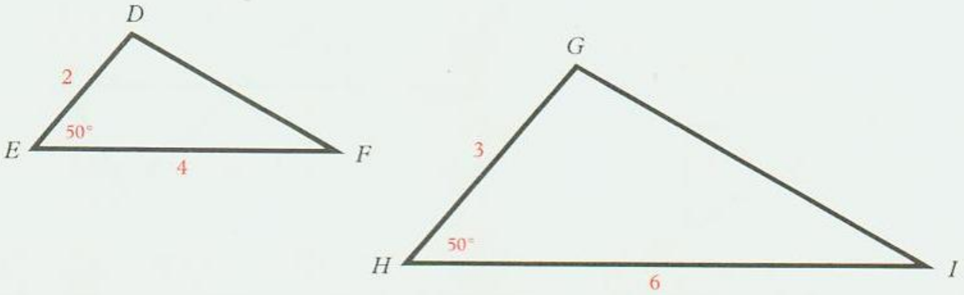


Since

$$\frac{40}{200} = \frac{32}{160} = \frac{27}{135} = \frac{1}{5},$$

Theorem 9-12 answers the question. Yes, $\triangle ABF \sim \triangle A'B'F'$.

There is another way of showing two triangles are similar. $\triangle DEF$ and $\triangle GHI$ were constructed so that $\frac{DE}{GH} = \frac{EF}{HI}$ and $\angle E \cong \angle H$.



These conditions imply that $\angle F \cong \angle I$ and $\angle D \cong \angle G$. This example suggests the following theorem called the SAS Similarity Theorem.

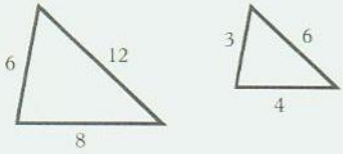
Theorem 9-13 **SAS Similarity Theorem.** If two triangles have an angle of one triangle congruent to an angle of the other triangle, and if the corresponding sides including the angle are proportional, then the triangles are similar.

EXERCISES

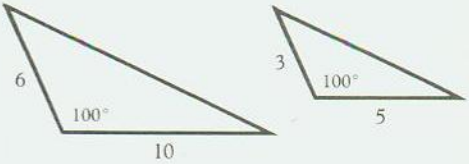
A.

In exercises 1–4 use the given information to decide if the triangles are similar. Base your decision on the measurements given rather than the shape of the drawings. If the triangles are similar, state whether the AA, SAS, or SSS Similarity Theorem applies.

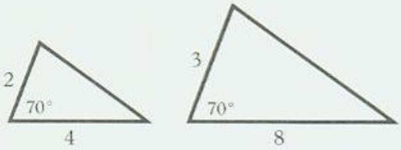
1.



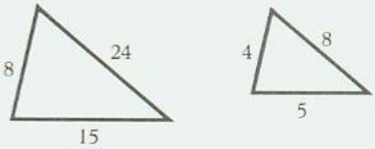
2.



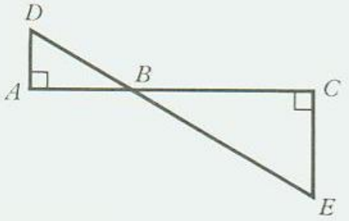
3.



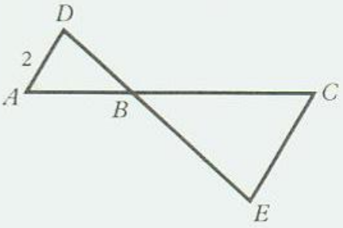
4.



5. If $AB = 4$ and $BC = 7$, find $\frac{AD}{CE}$.

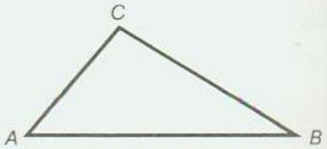


6. If point B divides $\overline{AC}$ and $\overline{DE}$ into thirds, how long is $\overline{CE}$?



Activity

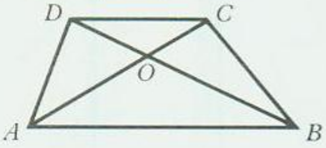
1. Construct a $\triangle ABC$ with sides 10 cm, 13 cm, and 17 cm long. Find the point of intersection of the perpendicular bisectors of the sides.
2. This point is the center of the circle that passes through the three vertices. This circle is called the circumscribed circle. Construct this circle.



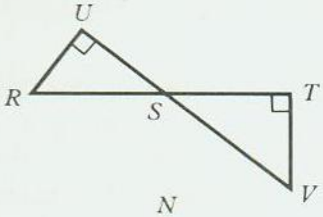
3. Then construct the nine-point circle, as described on page 246.
4. How does the radius of the circumscribed circle compare to the radius of the nine-point circle?

B.

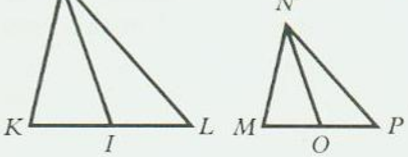
7. **Given:** Trapezoid $ABCD$, $\overline{AB} \parallel \overline{CD}$
Prove: $\triangle AOB \sim \triangle COD$



8. **Given:** $\overline{RU} \perp \overline{UV}$, $\overline{TV} \perp \overline{RT}$
Prove: $\triangle RSU \sim \triangle VST$



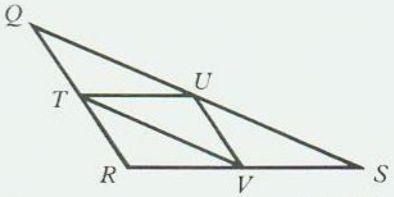
9. **Given:** $\triangle JKL \sim \triangle NMP$
$\overline{JI}$ and $\overline{NO}$ are medians.
Prove: $\dfrac{JI}{NO} = \dfrac{JK}{NM}$



(Ex. 9)

10. **Given:** T, U, and V are midpoints.
Prove: $\triangle QRS \sim \triangle VUT$

11. Draw a counterexample for this statement.
If two sides of one triangle are proportional to two sides of another triangle and an angle of the first triangle is congruent to an angle of the second triangle, the triangles are similar.



(Ex. 10)

C.

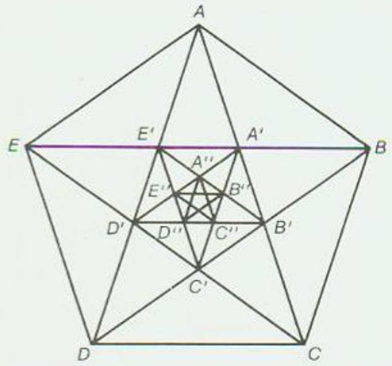
12. **Prove:** The corresponding diagonals of two similar quadrilaterals are in the same ratio as the corresponding sides.

13. **Prove:** Two isosceles triangles with congruent vertex angles are similar.

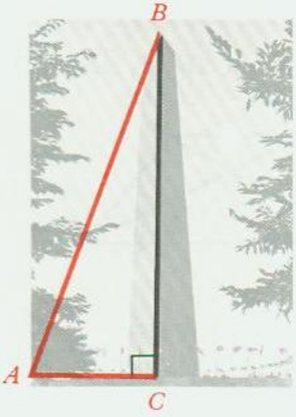
14. Draw a triangle given two acute angles and the length of the altitude to the included side.

PROBLEM SOLVING

Identify as many different-shaped figures as possible for which you can find another figure that is similar.

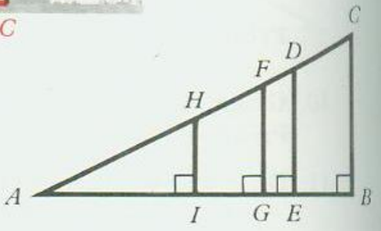


9–7 Trigonometric Ratios—An Application of Similar Triangles

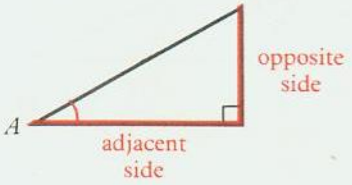


The heights of very tall buildings can be determined with the aid of ratios in a right triangle.

If distance AC is known and if the angle measure of $\angle A$ is known, then height BC can be calculated using a method studied in this lesson.



In this figure $\triangle ABC \sim \triangle AED \sim \triangle AGF \sim \triangle AIH$. Therefore, ratios of corresponding sides are equal.

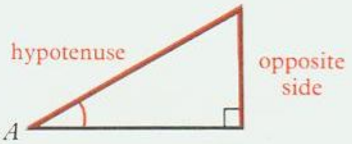


The ratios $\frac{BC}{AB}, \frac{DE}{AE}, \frac{FG}{AG}$, and $\frac{HI}{AI}$ in the figure above are all equal. These ratios are associated with $\angle A$ and are called the *tangent* of $\angle A$. This is abbreviated *tan A*.

Definition 9-4

The **tangent** of an acute angle of a right triangle is the ratio

$$\frac{\text{length of opposite side}}{\text{length of adjacent side}}.$$

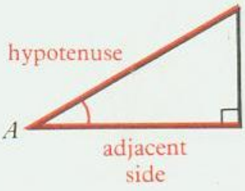


The ratios $\frac{BC}{AC}, \frac{ED}{AD}, \frac{GF}{AF}$, and $\frac{IH}{AH}$ are all equal. These ratios associated with $\angle A$ are called the *sine* of $\angle A$. This is abbreviated *sin A*.

Definition 9-5

The **sine** of an acute angle of a right triangle is the ratio

$$\frac{\text{length of opposite side}}{\text{length of hypotenuse}}.$$



The ratios $\frac{AB}{AC}, \frac{AE}{AD}, \frac{AG}{AF}$, and $\frac{AI}{AH}$ are all equal. These ratios associated with $\angle A$ are called the *cosine* of $\angle A$. This is abbreviated *cos A*.

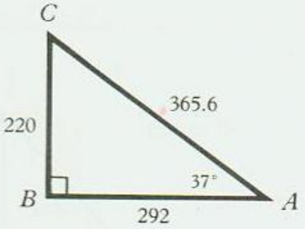
Definition 9-6

The **cosine** of an acute angle of a right triangle is the ratio

$$\frac{\text{length of adjacent side}}{\text{length of hypotenuse}}.$$

Example 1

From the figure at the right we can see that



$$\tan 37^\circ = \frac{220}{292} \doteq 0.7534,$$

$$\sin 37^\circ = \frac{220}{365.6} \doteq 0.6018,$$

$$\cos 37^\circ = \frac{292}{365.6} \doteq 0.7986.$$

These trigonometric ratios can be found for various angles using either a table of values, as shown here, or by using a calculator that has the trigonometric functions.

$m\angle A$ in degrees	tan A	sin A	cos A
1	.0175	.0175	.9998
2	.0349	.0349	.9994
3	.0524	.0523	.9986
4	.0699	.0698	.9976
5	.0875	.0872	.9962
6	.1051	.1045	.9945
7	.1228	.1219	.9925
8	.1405	.1392	.9903
9	.1584	.1564	.9877
10	.1763	.1736	.9848
11	.1944	.1908	.9816
12	.2126	.2079	.9781
13	.2309	.2250	.9744
14	.2493	.2419	.9703
15	.2679	.2588	.9659
16	.2867	.2756	.9613
17	.3057	.2924	.9563
18	.3249	.3090	.9511
19	.3443	.3256	.9455
20	.3640	.3420	.9397
21	.3839	.3584	.9336
22	.4040	.3746	.9272
23	.4245	.3907	.9205
24	.4452	.4067	.9135
25	.4663	.4226	.9063
26	.4877	.4384	.8988
27	.5095	.4540	.8910
28	.5317	.4695	.8829
29	.5543	.4848	.8746
30	.5774	.5000	.8660
31	.6009	.5150	.8572
32	.6249	.5299	.8480
33	.6494	.5446	.8387
34	.6745	.5592	.8290
35	.7002	.5736	.8192
36	.7265	.5878	.8090
37	.7536	.6018	.7986
38	.7813	.6157	.7880
39	.8098	.6293	.7771
40	.8391	.6428	.7660
41	.8693	.6561	.7547
42	.9004	.6691	.7431
43	.9325	.6820	.7314
44	.9657	.6947	.7193
45	1.0000	.7071	.7071

Example 2

From the table of approximate values we see that

$$\tan 42^\circ = 0.9004,$$
$$\sin 42^\circ = 0.6691,$$
$$\cos 42^\circ = 0.7431.$$

APPLICATION

A person 1000 feet from the base of the Washington Monument finds the measure of $\angle A$ to be about 29°. About how high is the monument?

$$\tan 29^\circ = \frac{x}{1000} \quad \text{or} \quad x = 1000 \times \tan 29^\circ$$
$$= 1000 \times .5543 = 554.3 \text{ feet}$$

EXERCISES

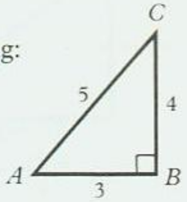
A.

1. Complete the following:

$\tan A = \underline{?}$

$\sin A = \underline{?}$

$\cos A = \underline{?}$

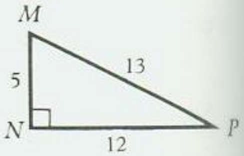


2. Complete the following:

$\tan M = \underline{?}$

$\sin P = \underline{?}$

$\cos P = \underline{?}$



For exercises 3–14, use the trigonometric tables.

3. $\sin 17^\circ = \underline{?}$ **4.** $\cos 43^\circ = \underline{?}$ **5.** $\tan 21^\circ = \underline{?}$

6. $\cos 13^\circ = \underline{?}$ **7.** $\tan 35^\circ = \underline{?}$ **8.** $\sin 37^\circ = \underline{?}$

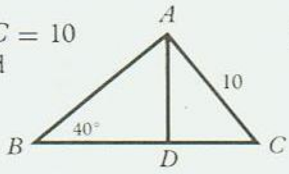
Find $m\angle A$ to the nearest degree given the following approximate measures:

9. $\tan \angle A = 0.7536.$ **10.** $\cos \angle A = 0.9985.$ **11.** $\sin \angle A = 0.2925.$

12. $\cos \angle A = 0.8290.$ **13.** $\sin \angle A = 0.0699.$ **14.** $\tan \angle A = 0.9658.$

B.

15. $\triangle ABC$ has $AB = AC = 10$ and $m\angle B = 40.$ Find the length of $\overline{AD}.$



16. $\triangle ABC$ has $AB = AC = 10$ and $m\angle B = 40.$ Find the length of $\overline{BD}.$

(Exs. 15, 16)

Activity

Use a calculator to determine the following.

1. a. $(\sin 57^\circ)^2 + (\cos 57^\circ)^2 = \underline{?}$ **b.** $(\sin 43^\circ)^2 + (\cos 43^\circ)^2 = \underline{?}$

c. $(\sin 9^\circ)^2 + (\cos 9^\circ)^2 = \underline{?}$ **d.** $(\sin 24^\circ)^2 + (\cos 24^\circ)^2 = \underline{?}$

e. What do you guess about $\sin^2 x + \cos^2 x$ for all x?

2. a. Compare $\frac{\sin 47^\circ}{\cos 47^\circ}$ and $\tan 47^\circ$. **b.** Compare $\frac{\sin 71^\circ}{\cos 71^\circ}$ and $\tan 71^\circ$.

c. Compare $\frac{\sin 33^\circ}{\cos 33^\circ}$ and $\tan 33^\circ$. **d.** Compare $\frac{\sin 66^\circ}{\cos 66^\circ}$ and $\tan 66^\circ$.

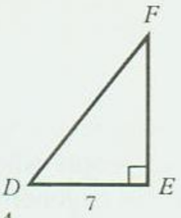
e. What do you guess about $\frac{\sin x}{\cos x}$ and $\tan x$?

17. $\sin F = \frac{7}{25}$

$FD = \underline{?}$

$FE = \underline{?}$

$\tan D = \underline{?}$

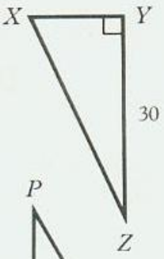


18. $\tan Z = \frac{8}{15}$

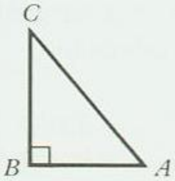
$XY = \underline{?}$

$XZ = \underline{?}$

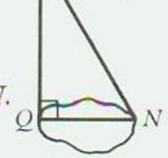
$\cos X = \underline{?}$



19. Suppose $AB = 3$, $BC = 4$, and $AC = 5$. Use the trig tables or a calculator to determine $m\angle A$ and $m\angle C$ to the nearest degree.

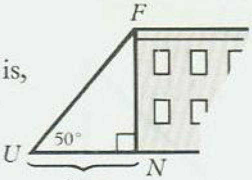


20. Suppose you want to find the distance across the pond QN. You measure to find that $PQ = 50$ m and determine $m\angle P = 44$. Find QN.

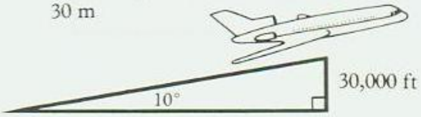


C.

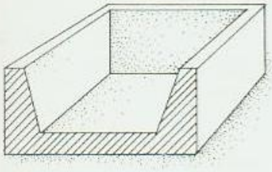
21. Suppose you want to find the height of the building, that is, FN. The $m\angle FUN = 50$ at a distance 30 meters from N. What is the length of FN?

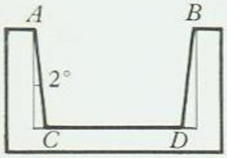


22. If an airplane takes off and climbs at a steady rate of 10° until an altitude of 30,000 feet is reached, what is the ground distance covered?



23. Melted gold is poured into a mold to form a brick. A box-like mold with inner slanting sides is used. The "draft angle" is the degree of slant of the walls. The draft angle is 2° and $BD = 6$ cm. How much wider is AB than CD?

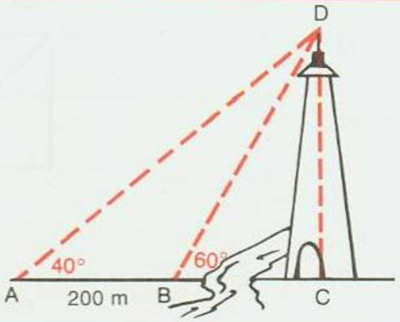




PROBLEM SOLVING

Suppose you want to find the height (DC) of a tower but you cannot measure distances AC and BC directly.

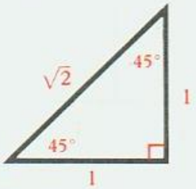
If $m\angle A = 40$, $m\angle DBC = 60$, and $AB = 200$ m, find height DC.



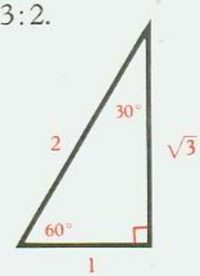
9–8 Trigonometric Ratios of Special Angles

Just as the 30°, 45°, and 60° angles are important to people who do drafting work, they are also important special angles in trigonometry. It is often helpful to know the trigonometric ratios of these angles without referring to tables of values or to a handheld calculator.

From Theorem 7–3 we conclude that the side lengths of a 45°-45°-90° triangle are in a ratio of $1:1:\sqrt{2}$.



From Theorem 7–4 we conclude that the side lengths of a 30°-60°-90° triangle are in a ratio of $1:\sqrt{3}:2$.



This table shows the trigonometric ratios for these special angles.

	30°	60°	45°
tan	$\frac{\sqrt{3}}{3}$	$\sqrt{3}$	1
sin	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$
cos	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$

Example 1

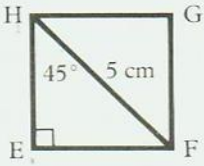
The diagonal of a square is 5 cm. Find the length of a side.

$$\sin \angle EHF = \frac{EF}{5}$$

$$\sin 45^\circ = \frac{EF}{5}$$

$$\frac{\sqrt{2}}{2} = \frac{EF}{5}$$

$$EF = \frac{5\sqrt{2}}{2}\text{ cm}$$



Example 2

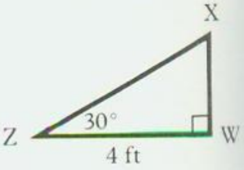
In the triangle shown find XW and XZ.

$$\tan \angle Z = \frac{XW}{4}$$

$$\tan 30^\circ = \frac{XW}{4}$$

$$\frac{\sqrt{3}}{3} = \frac{XW}{4}$$

$$XW = \frac{4\sqrt{3}}{3}\text{ ft}$$



Since XZ is $2XW$, then $XZ = \frac{8\sqrt{3}}{3}$ ft.

EXERCISES

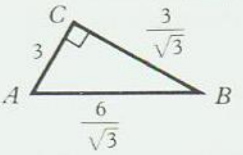
A.

For exercises 1–6, complete the statement correctly without referring to the table on page 331.

1. $\sin 45° = \underline{?}$ **2.** $\cos 30° = \underline{?}$ **3.** $\tan 60° = \underline{?}$

4. $\cos 60° = \underline{?}$ **5.** $\sin 30° = \underline{?}$ **6.** $\tan 45° = \underline{?}$

For exercises 7–14, the right triangle shown is not drawn accurately. Accept the side lengths as shown. Complete the following statements correctly.



(Exs. 7–14)

7. $\sin A = \underline{?}$ **8.** $\cos A = \underline{?}$ **9.** $\tan A = \underline{?}$

10. $\cos B = \underline{?}$ **11.** $\tan B = \underline{?}$ **12.** $\sin B = \underline{?}$

13. $m\angle A = \underline{?}$ **14.** $m\angle B = \underline{?}$

B.

For exercises 15–18, evaluate each given expression. Identify those pairs of expressions that are equal. In each problem assume that $m\angle A = 30$.

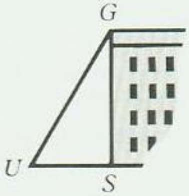
15. $2 \sin A$, $2 \sin A \cos A$, $\sin 2A$

16. $\cos 2A$, $(\cos A)^2 - (\sin A)^2$, $2(\cos A)^2 - 1$

17. $\tan 2A$, $2 \tan A$, $\dfrac{2 \tan A}{1 - (\tan A)^2}$

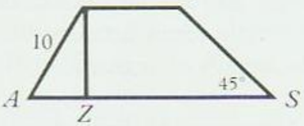
18. $2 \cos A$, $1 - 2(\sin A)^2$, $\cos 2A$

C.



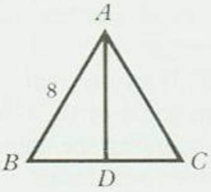
19. If $m\angle GUS = 30$ and $US = 50$ m, how high is the building?

20. Quadrilateral $EASY$ is an isosceles trapezoid with $EA = SY$. If $EA = 10$ and $m\angle EAS = 45$, find the length of the altitude $\overline{EZ}$.



21. $\triangle ABC$ is equilateral. Find the length of the altitude $\overline{AD}$.

22. Show that, if $\triangle ABC$ has right angle $\angle C$, then $(\sin A)^2 + (\cos A)^2 = 1$.



(Ex. 21)

Important Ideas—Chapter 9

Terms

Proportion (p. 304)	Tangent of an acute angle (p. 330)
Similar polygons (p. 313)	Sine of an acute angle (p. 330)
Geometric mean (p. 322)	Cosine of an acute angle (p. 330)

Postulate

AAA Similarity Postulate (p. 316)

Theorems

9-1 If $\frac{a}{b} = \frac{c}{d}$ then $a \times d = b \times c$.

9-2 If $\frac{a}{b} = \frac{c}{d}$, then $\frac{a+b}{b} = \frac{c+d}{d}$.

9-3 If $\frac{a}{b} = \frac{c}{d}$, then $\frac{a-b}{b} = \frac{c-d}{d}$.

9-4 If $\frac{a}{b} = \frac{c}{d}$, then $\frac{a}{c} = \frac{b}{d}$.

9-5 If $a \times d = b \times c$, then $\frac{a}{b} = \frac{c}{d}$.

9-6 **Side-Splitting Theorem.** If a line is parallel to one side of a triangle and intersects the other two sides, then it divides the two sides proportionally.

9-7 If a line intersects two sides of a triangle and divides them proportionally, then the line is parallel to the third side.

9-8 **AA Similarity Theorem.** If two angles of one triangle are congruent to two angles of another triangle, then the two triangles are similar.

9-9 Two right triangles are similar if an acute angle of one triangle is congruent to an acute angle of the other triangle.

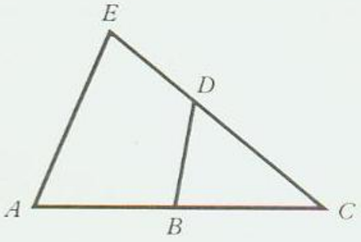
9-10 In a right triangle, the length of the altitude to the hypotenuse is the geometric mean between the lengths of the two segments of the hypotenuse.

9-11 Given a right triangle and the altitude to the hypotenuse, each leg is the geometric mean between the length of the hypotenuse and the length of segment of the hypotenuse adjacent to the leg.

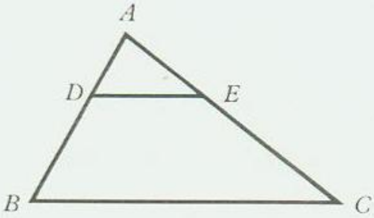
9-12 **SSS Similarity Theorem.** If three sides of one triangle are proportional to the three sides of another triangle, then the triangles are similar.

9-13 **SAS Similarity Theorem.** If two triangles have an angle of one triangle congruent to an angle of the other triangle, and if the corresponding sides including the angle are proportional, then the triangles are similar.

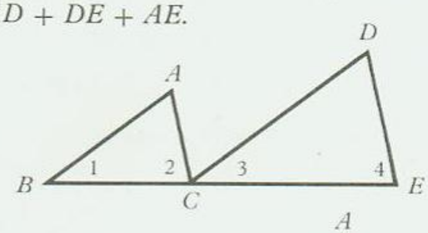
Chapter 9—Review



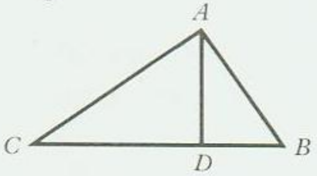
1. Indicate whether the following statements are true or false.
   a. If $\overline{BD} \parallel \overline{AE}$, then $\frac{CD}{DE} = \frac{CB}{BA}$.
   b. If $\frac{CD}{CE} = \frac{CB}{CA}$, then $\overline{BD} \parallel \overline{AE}$.
   c. If $CD = 4$, $DE = 3$, $BC = 8$, then $AB = 6$.
   d. If $\angle A \cong \angle DBC$, then $\triangle EAC \sim \triangle DBC$.
   e. If $CD = DE = AB = BC$, then $\triangle EAC \sim \triangle DBC$.



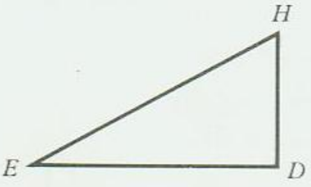
2. Assume $\overline{DE} \parallel \overline{BC}$.
   a. $AD = 4$, $BD = 6$, $AE = 5$. Find AC.
   b. $AB = 10$, $BD = 7$, $AC = 12$. Find AE.
   c. $AD = 3$, $BD = 4$, $BC = 6$. Find DE.
   d. $AB = BC = AC = 6$, $AD = 2$. Find $AD + DE + AE$.



3. **Given:** $\overline{AB} \parallel \overline{CD}$
   $\overline{AC} \parallel \overline{DE}$
   Prove: $AB \cdot CE = BC \cdot DC$

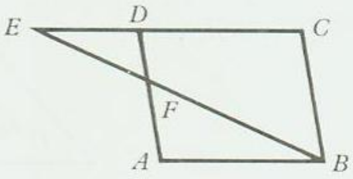


4. Assume $m\angle CAB = 90$ and $\overline{AD} \perp \overline{BC}$.
   a. If $AB = 8$ and $BC = 12$, find BD.
   b. If $AC = 6$ and $DC = 4$, find BC.
   c. If $BD = 4$ and $AD = 6$, find DC.



5. Assume $m\angle D = 90$.
   a. If $\sin \angle E = \frac{8}{17}$, find $\tan \angle E$.
   b. If $\cos \angle H = \frac{9}{41}$, find $\cos \angle E$.
   c. If $\tan \angle E = \frac{4}{3}$, find $\cos \angle H$.

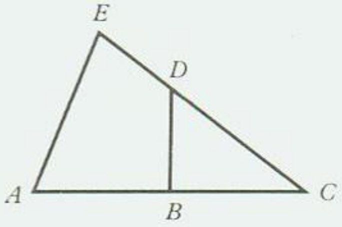
6. If a 20-foot tree casts a shadow of 45 feet, how long a shadow would a 30-foot tree cast?



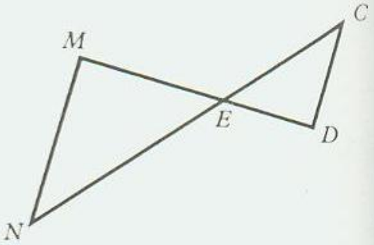
7. **Given:** Figure $ABCD$ is a parallelogram.
   Prove: **a.** $\triangle ECB \sim \triangle EDF$
   b. $\triangle EDF \sim \triangle BAF$
   c. $\triangle ECB \sim \triangle BAF$

Chapter 9—Test

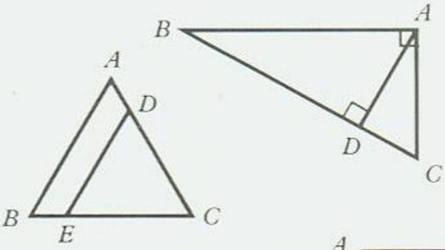
1. Indicate whether the following statements are true or false.
   - **a.** If $\overline{DB} \parallel \overline{AE}$, then $CD \cdot DE = BC \cdot AB$
   - **b.** If $\frac{EC}{DC} = \frac{AC}{BC}$, then $\triangle AEC \sim \triangle BDC$.
   - **c.** If $\triangle AEC \sim \triangle BDC$, then $\frac{EC}{ED} = \frac{AC}{AB}$.
   - **d.** If $\frac{EC}{DC} = \frac{AC}{BC}$, then $\triangle ACE \sim \triangle BCD$.
   - **e.** If $DC \cdot AB = BC \cdot ED$, then $\overline{BD} \parallel \overline{AE}$.



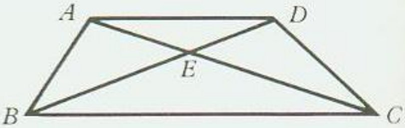
2. Assume $\overline{MN} \parallel \overline{CD}$.
   - **a.** If $ME = 2ED$, find $\frac{NE}{EC}$.
   - **b.** If $ME = 4$, $NE = 5$, and $EC = 3$, find ED.
   - **c.** If $ME = 4$, $DE = 2$, and $NC = 9$, find EC.
   - **d.** If $MN = ME = NE = 6$ and $CE = 4$, find $CE + CD + DE$.



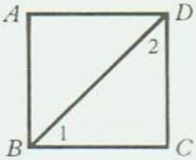
3. Assume $\triangle ABC$ and $\triangle ABD$ are right triangles.
   - **a.** If $AC = 3$ and $AB = 4$, find AD.
   - **b.** If $AB = 12$ and $BC = 13$, find DC.
   - **c.** If $BD = 9$ and $BC = 15$, find AD.
   - **d.** If $AC = 20$ and $BC = 40$, find DC.
4. **Given:** $AB = AC$
   $DE = DC$
   Prove: $\triangle ABC \sim \triangle DEC$



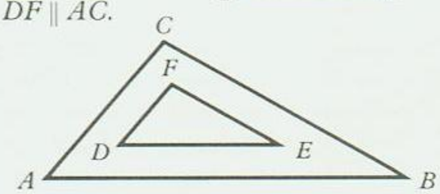
5. **Given:** $ABCD$ is a trapezoid.
   Prove: $DE \cdot EC = AE \cdot BE$



6. $ABCD$ is a square with diagonal $\overline{BD}$.
   - **a.** Find $\tan \angle 1$.
   - **b.** Find $\cos \angle 2$.
   - **c.** Does $\cos \angle 1 = \sin \angle 1$?



7. **Given:** $\overline{DE} \parallel \overline{AB}$, $\overline{EF} \parallel \overline{BC}$, $\overline{DF} \parallel \overline{AC}$.
   Prove: $\triangle DEF \sim \triangle ABC$



Problem Solving Techniques

Work Backwards

Sometimes when solving problems it is helpful to work backwards as well as forwards. Ask yourself, "What information do I need in order to reach the conclusion I want?"

Example

Consider the problem on the right. Use the questions and answers below to help solve the problem.

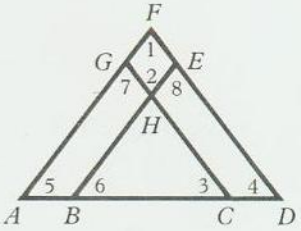
Given: $\angle 1 \cong \angle 2$, $\angle 3 \cong \angle 4$
$\overline{AB} \cong \overline{CD}$
Prove: $\triangle AGC \cong \triangle BED$

a. **Question:** How can I prove triangles congruent?

b. **Question:** Which segments are congruent?

c. **Question:** Which sides of the triangles can I prove congruent?

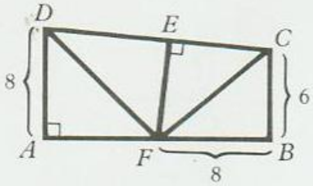
d. **Question:** Which angles of the triangles can I prove congruent?



PROBLEMS

Solve the problems by working backwards.

1. **Given:** $\overline{AD} \perp \overline{AB}$, $\overline{CB} \perp \overline{AB}$
$\overline{EF}$ bisects and is perpendicular to $\overline{DC}$.
Find: AF



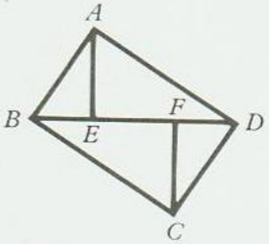
a. What kind of triangle is $\triangle ADF$?

b. Can I use the Pythagorean Theorem?

c. In order to find AF, which side of $\triangle ADF$ do I need to find?

d. What do I know about $\overline{DF}$?

2.



Given: $m\angle BAD = m\angle DCB = 90$,
$\overline{AE} \perp \overline{BD}$, $\overline{FC} \perp \overline{BD}$,
$AB = CD = 9$, $AD = BC = 12$
Find: EF

3. Suppose $\angle 1 \cong \angle 2$ and $\angle 3 \cong \angle 4$. What must be the measure of $\angle C$ so that $\overline{DE} \parallel \overline{GF}$?

a. Suppose $\overline{DE} \parallel \overline{GF}$. Then, $m\angle DEF + m\angle GFE = 180$.

b. Let $m\angle 1 = m\angle 2 = x$ and $m\angle 3 = m\angle 4 = y$. Find an expression for $m\angle 2 + m\angle 4$ and then for $m\angle C$.

c. Reverse this reasoning to prove that your answer assures that $\overline{DE} \parallel \overline{GF}$.